Apply Kalman Filter in Financial Time Series
Final Project for EE616 Signal Detection & Estimation

Xingzhong Xu

Department of Electrical & Computer Engineering
Stevens Institute of Technology

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Financial time series are well-known non-stationary.

There’s no perfect prediction model for such time series.

A fundamental assumption is that the underlying series are driven by some hidden control or variables.

A good approximate model should,
- demonstrates the hidden effects (state-space model)
- provide a good prediction performance (mean square error)
- computationally efficient (recursive filtering)

In this project, I will use dynamic state-space system to model the financial time series, and then use Kalman filter to efficiently make prediction.
Review Kalman Filter

- Under a Gaussian-Markov state model \((u[n] \sim \mathcal{N}(0, Q))\)
  \[ s[n] = As[n-1] + Bu[n] \]

- and Bayesian linear observation model \((w[n] \sim \mathcal{N}(0, C[n]))\)
  \[ x[n] = H[n]s[n] + w[n] \]

- a Kalman filter is a recursive (prediction & correction only use present input \(x[n]\) and previous output \(\hat{s}[n-1|n-1]\), \(K[n]\) is Kalman Gain \(^1\))
  \[ \hat{s}[n|n-1] = A\hat{s}[n-1|n-1] \]
  \[ \hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - H[n]\hat{s}[n|n-1]) \]

- MMSE estimator (\(M\) are minimum mean square error matrix).
  \[ M[n|n-1] = AM[n-1|n-1]A^T + BQB^{-1} \]
  \[ M[n|n] = (I - K[n]H[n])M[n|n-1] \]

\(^1\)
\[
\]
Basic Model

In finance, compare to the assets price $p$, the rate of return $r$ tend to behavior more stationary. We denote $r$ as

$$r[n] = \log(p[n]) - \log(p[n - 1])$$

Although the true value of $r[n]$ is unknown, we could always observer it in noise market by,

$$R[n] = r[n] + w[n] \quad w[n] \sim \mathcal{N}(0, \sigma_w^2)$$

In this project, I will analyzing two models with different assumptions as follows

- $r$ is constant.
- $r$ is mean reverting.
Constant $r$

We firstly assume the $r$ is constant, then,

$$r[n] = r[n - 1] + u$$

We further assume the observation and process noises are WSS ($u \sim \mathcal{N}(0, \sigma_u^2)$, $w \sim \mathcal{N}(0, \sigma_w^2)$) and $\sigma_u^2 \ll \sigma_w^2$.

Recall the Kalman filter discussion, we have

$$\hat{r}[n|n - 1] = \hat{r}[n - 1|n - 1]$$
$$M[n|n - 1] = M[n - 1|n - 1] + \sigma_u^2$$
$$K[n] = \frac{M[n|n - 1]}{\sigma_w^2 + M[n|n - 1]}$$
$$\hat{r}[n|n] = \hat{r}[n|n - 1] + K[n](R[n] - \hat{r}[n|n - 1])$$
$$M[n|n] = (1 - K[n])M[n|n - 1]$$
In the above model, we assume $\sigma_u$, $\sigma_w$ and $\mu$ are constant parameters.

Now we estimate them from real data.

Recall the Gaussian Linear assumption and $\sigma_u^2 \ll \sigma_w^2$,

\[
\begin{align*}
R & \sim \mathcal{N}(r, \sigma_w^2 I) \\
r & \sim \mathcal{N}(\mu, \sigma_u^2 I) \\
R & \sim \mathcal{N}(\mu, (\sigma_w^2 + \sigma_u^2) I) \\
R & \sim \mathcal{N}(\mu, \sigma_w^2 I)
\end{align*}
\]

The MLE of $\gamma = \begin{bmatrix} \sigma_w & \mu \end{bmatrix}^T$ is given by,

\[
\arg \max_{\gamma} L(\gamma | R)
\]
Constant $r$ - Parameter Estimation

\[ R \sim \mathcal{N}(\mu, \sigma_w^2 I) \]

\[
\log L(\gamma | R) = \log f(R, \gamma)
\]

\[
= \log \frac{\exp\left(-\frac{\sum_n (R[n] - \mu)^2}{2\sigma_w^2}\right)}{(2\pi \sigma_w)^{N/2}}
\]

\[
= \frac{N}{2} \log(2\pi \sigma_w^2) + \frac{\sum_n (R[n] - \mu)^2}{2\sigma_w^2}
\]

\[
\frac{\partial \log L(\gamma | R)}{\partial \mu} = \frac{\sum_n (R[n] - \mu)}{\sigma_w^2}
\]

(set to 0)

\[
\hat{\mu} = \frac{1}{N} \sum_n R[n]
\]

\[
= \bar{R}[n] \quad \text{(MLE of } \mu)\]
\[
\frac{\partial \log L(\gamma|R)}{\partial \sigma_w^2} = \frac{N}{2\sigma_w^2} - \sum_n (R[n] - \mu)^2 \quad \text{(set to 0)}
\]

\[
\hat{\sigma}_w^2 = \frac{1}{N} \sum_n (R[n] - \mu)^2
\]

\[
= \frac{1}{N} \sum_n (R[n] - \bar{R}[n])^2 \quad \text{(MLE of } \sigma_w^2 \text{)}
\]

\[
\hat{\gamma} = \left[ \frac{1}{N} \sum_n (R[n] - \bar{R}[n])^2 \right]
\]
Exxon Mobil Corporation (NYSE:XOM) historical daily price and return from 2008-01-23 to 2012-04-26.

Use first 80% data to find the MLE of $\gamma = \begin{bmatrix} 0.003936\% & 0.0435\% \end{bmatrix}^T$. 
Use latest 20% data to recursively evaluate the $\hat{r}[n|n-1]$. 

![Graph showing daily price and predicted price, as well as daily return and predicted return.](image)
Mean-reverting Model

- Now we relax $r$’s constant assumption.
- Let us assume $\mathbb{E}(r_n) = \mu$, and $r$ is mean-reverting.

$$r_n - r_{n-1} = \alpha(\mu - r_{n-1}) + u$$

- Then the state space model will be given by

$$r_n = (1 - \alpha)r_{n-1} + \alpha\mu + u$$

$$\mathbb{E}(r) = \mu$$

$$\text{var}(r) = \frac{\sigma_u^2}{2\alpha - \alpha^2}$$

- The observation model will be given by

$$R_n = r_n + w$$
Mean-reverting Model - 100 sample simulation

\[ \mu = 0.1 \quad \sigma_u^2 = 0.1 \]
In the above model, we assume the $\alpha$, $\sigma_w$, $\sigma_u$ and $\mu$ are unknown constant parameters.

According to the linear Gaussian assumption,

$$R_n = r_n + w$$
$$= (1 - \alpha) r_{n-1} + \alpha \mu + u + w$$
$$= (1 - \alpha)(R_{n-1} - w) + \alpha \mu + u + w$$
$$= (1 - \alpha)R_{n-1} + \alpha \mu + u + \alpha w$$

which shows $R_n$ is an autoregressive process AR(1).

We would like to obtain the MLE of $\gamma = [\alpha \ \sigma_w^2 \ \sigma_u^2 \ \mu]^T$, 

$$\arg \max_{\gamma} L(\gamma|R)$$
Mean-reverting Model - Conditional MLE

\[ R_n | R_{n-1} \sim \mathcal{N}((1 - \alpha)R_{n-1} + \alpha\mu, \sigma_u^2 + \alpha^2\sigma_w^2) \]

\[ f(R_n | R_{n-1}, \gamma) = \frac{1}{\sqrt{2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)}} \exp\left(-\frac{(R_n - (1 - \alpha)R_{n-1} - \alpha\mu)^2}{2(\sigma_u^2 + \alpha^2\sigma_w^2)}\right) \]

\[ \log(f(R_n | R_{n-1}, \gamma)) = -\log(2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)) - \frac{(R_n - (1 - \alpha)R_{n-1} - \alpha\mu)^2}{2(\sigma_u^2 + \alpha^2\sigma_w^2)} \]
Recall $R$ is a stationary AR(1) process, we can assume $R_1$ as,

$$E[R_1] = \mu \quad \text{var}[R_1] = \frac{\sigma_u^2 + \alpha^2 \sigma_w^2}{2\alpha - \alpha^2}$$

$$R_1 \sim \mathcal{N}\left(\mu, \frac{\sigma_u^2 + \alpha^2 \sigma_w^2}{2\alpha - \alpha^2}\right)$$

$$f(R_1, \gamma) = \left(\frac{2\pi(\sigma_u^2 + \alpha^2 \sigma_w^2)}{2\alpha - \alpha^2}\right)^{-1/2} \exp\left(-\frac{(R_1 - \mu)^2(2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2 \sigma_w^2)}\right)$$

$$\log(f(R_1, \gamma)) = -\frac{1}{2} \log\left(\frac{2\pi(\sigma_u^2 + \alpha^2 \sigma_w^2)}{2\alpha - \alpha^2}\right) - \frac{(R_1 - \mu)^2(2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2 \sigma_w^2)}$$
Mean-reverting Model - Exact MLE

\[ f(R_n, \ldots, R_1 | \gamma) = f(R_1, \gamma) \prod_{t=2}^{n} f(R_n | R_{n-1}, \gamma) \]

\[ \log L(\gamma | R) = \log f(R_1, \gamma) + \sum_{t=2}^{n} \log f(R_t | R_{t-1}, \gamma) \]

\[ = -\frac{1}{2} \log \left( \frac{2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)}{2\alpha - \alpha^2} \right) - \frac{(R_1 - \mu)^2(2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2\sigma_w^2)} \]

\[ - \sum_{t=2}^{n} \left( \log(2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)) - \frac{(R_t - (1 - \alpha)R_{t-1} - \alpha\mu)^2}{2(\sigma_u^2 + \alpha^2\sigma_w^2)} \right) \]

\[ = \log(2\alpha - \alpha^2) \left( \frac{2\alpha - \alpha^2}{2} \right) - \frac{n}{2} \log(2\pi(\sigma_u^2 + \alpha^2\sigma_w^2)) - \frac{(R_1 - \mu)^2(2\alpha - \alpha^2)}{2(\sigma_u^2 + \alpha^2\sigma_w^2)} \]

\[ - \frac{1}{2(\sigma_u^2 + \alpha^2\sigma_w^2)} \sum_{t=2}^{n} (R_t - (1 - \alpha)R_{t-1} - \alpha\mu)^2 \]
Mean-reverting Model - Kalman filter

Notice that the log-likelihood function $\log L(\gamma|\mathbf{R})$ is a non-linear function, so there’s no exact analytical solution for MLE $\hat{\gamma}$. Here we use numerical method,

$$\arg \max_{\gamma} \log L(\gamma|\mathbf{R})$$

We then use MLE $\gamma$ to configure a Kalman filter.

$$\hat{r}[n|n-1] = (1-\hat{\alpha})\hat{r}[n-1|n-1] + \hat{\alpha}\hat{\mu}$$
$$M[n|n-1] = (1-\hat{\alpha})^2 M[n-1|n-1] + \hat{\sigma}_u^2$$
$$K[n] = \frac{M[n|n-1]}{\hat{\sigma}_w^2 + M[n|n-1]}$$
$$\hat{r}[n|n] = \hat{r}[n|n-1] + K[n](\mathbf{R}[n] - \hat{r}[n|n-1])$$
$$M[n|n] = (1 - K[n])M[n|n-1]$$
Mean-reverting Model - 100 sample simulation

\[ \mu = 0.1 \quad \sigma^2_u = 0.1 \quad \sigma^2_w = 0.01 \quad \alpha = 1.4 \]
Exxon Mobil Corporation (NYSE:XOM) historical daily price and return from 2008-01-23 to 2012-04-26.

Use first 80% data to find the MLE of
\[ \gamma = \begin{bmatrix} 1.211 & 5.66 \times 10^{-4}\% & 4.15\% & 3.17 \times 10^{-3}\% \end{bmatrix}^T. \]
Mean-reverting Model - Application

- Use latest 20% data to recursively evaluate the $\hat{r}[n|n-1]$. 

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**Graph 1:**
- **X-axis:** Daily Price
- **Y-axis:** Predict Price
- **Legend:** Daily Return (blue plus symbols), Predict Return (red line)

**Graph 2:**
- **X-axis:** Daily Return
- **Y-axis:** Predict Return
- **Legend:** Daily Price (blue plus symbols), Predict Price (red line)
Mean-reverting model have better tracking error performance, especially when price change dramatically.
The financial time series in real applications are always non-stationary. So there’s no perfect model can fit them well.

I assume the daily return series are stationary, and thus using two state space model (constant and time-reverting) to model it separately.

Both models’ parameters were estimated (analytically or numerically) through maximizing its likelihood function.

Then based on the parameters, a configured Kalman filter is used to recursively predict and correct the underlying series.

Not surprisingly, a more complicated mean-reverting model have better prediction performance than the constant one.
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